MATH 1650: SECTION 6.3: PROPERTIES OF LOGARITHMS

RECALL: The inverse of $f(x) = b^x$ is $g(x) = \log_b(x)$. This means $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$. Writing these out, we get:

- $(g \circ f)(x) = x \iff g(f(x)) = x \iff g(b^x) = x \iff \log_b(b^x) = x$ for all real numbers, x.
- $(f \circ g)(x) = x \iff f(g(x)) = x \iff f(\log_b(x)) = x \iff b^{\log_b(x)} = x$ for all real numbers $x > 0$.

INVERSE PROPERTIES OF EXPONENTIAL AND LOGARITHM FUNCTIONS:

$$\log_b(b^x) = x \quad \text{for all real numbers } x \quad \textbf{and} \quad b^{\log_b(x)} = x \quad \text{for all } x > 0$$

EXAMPLE: Simplify the following expressions:

1. $\log_3(3^x)$
2. $\ln\left(e^{0.01t}\right)$
3. $10^{\log(-0.2)}$
4. $e^{x\ln(2)}$

OTHER PROPERTIES LOGARITHMS: Suppose M, N, and p are real numbers with $M > 0$ and $N > 0$.

- **PRODUCT RULE:** $\log_b(MN) = \log_b(M) + \log_b(N)$
- **QUOTIENT RULE:** $\log_b\left(\dfrac{M}{N}\right) = \log_b(M) - \log_b(N)$
- **POWER RULE:** $\log_b(M^p) = p\log_b(M)$

PROOF: Each of the rules of logarithms corresponds directly to a rule of exponents.

Suppose $\log_b(M) = m$, $\log_b(N) = n$, and p is a real number.

Then, $\log_b(M) = m$ means $b^m = M$ and $\log_b(N) = n$ means $b^n = N$. Hence:

- Using the **product rule** for exponents: $MN = b^m b^n = b^{m+n}$.

  Hence, $\log_b(MN) =$ the exponent you put on b to get $MN = m + n = \log_b(M) + \log_b(N)$ ✓

- Using the **quotient rule** for exponents: $\dfrac{M}{N} = \dfrac{b^m}{b^n} = b^{m-n}$.

  Hence, $\log_b\left(\dfrac{M}{N}\right) =$ the exponent you put on b to get $\dfrac{M}{N} = m - n = \log_b(M) - \log_b(N)$ ✓

- Using the **power rule** for exponents: $M^p = (b^m)^p = b^{pm}$.

  Hence, $\log_b(M^p) =$ the exponent you put on b to get $M^p = pm = p\log_b(M)$ ✓

EXAMPLE: 'Expand' the following using the properties of logarithms and simplify. That is, rewrite products as sums, quotients as differences, and powers as factors.

Assume when necessary that all quantities represent positive real numbers.

1. $\log_2 \left(\frac{8}{x}\right) =$

2. $\log_{0.1} \left(10x^2\right) =$

3. $\ln \left(\frac{3}{et}\right)^2 =$

4. $\log \sqrt[3]{\frac{100x^2}{yz^5}} =$

5. $\log_{117} \left(x^2 - 4\right) =$

   HINT: Factor $(x^2 - 4)$.

EXAMPLE: Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1) - \log_3(x+1)$

2. $\log(x) + 2\log(y) - \log(z)$

3. $4\log_2(x) + 3$

   HINT: Rewrite 3 as a logarithm base 2: $3 = \log_2\left(2^3\right) = \log_2(8)\ \ldots$

4. $-\ln(t) - \frac{1}{2}$

   HINT: Rewrite $\frac{1}{2}$ as a logarithm base e: $\frac{1}{2} = \ln\left(e^{\frac{1}{2}}\right) = \ln\sqrt{e}$.

CHANGE OF BASE FORMULAS: Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x. In particular, $a^x = e^{x \ln(a)}$.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$. In particular, $\log_a(x) = \dfrac{\ln(x)}{\ln(a)} = \dfrac{\log(x)}{\log(a)}$

EXAMPLE: Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a graphing utility, as appropriate.

1. 3^2 to base 10

2. 10^x to base e

3. $\log_4(5)$ to base e

4. $\ln(x)$ to base 10

HOMEWORK: Section 6.3: 1 - 41 odd.